Asymptotic values of entire meromorphic functions

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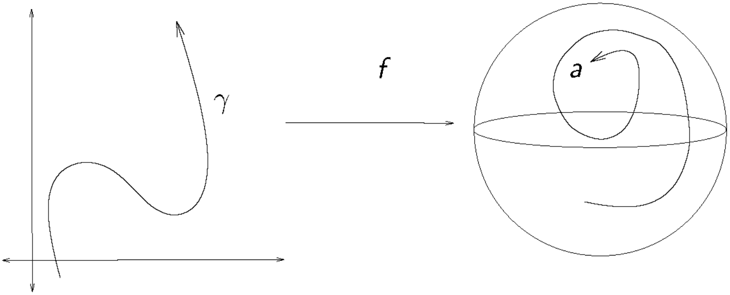
Definitions

Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ meromorphic, a is an asymptotic value for f if there exists a continuous curve γ such that

$$\lim_{\substack{z \to \infty \\ z \in \gamma}} f(z) = a \in \mathbb{C} \cup \{\infty\}$$



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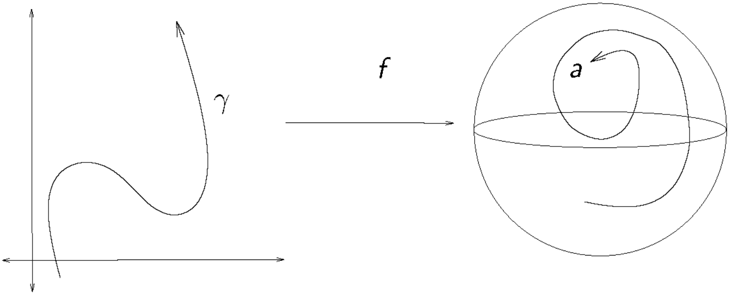

γ is an asymptotic path for a. $\mathrm{As}(f)$ denotes the set of asymptotic values of f. $\square$

Overview and known results

Theorem (Mazurkiewicz)

$f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ meromorphic then the set of asymptotic values of f is a (Suslin) analytic set of $\mathbb{C} \cup \{\infty\}$.

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- a continuous image of $\mathbb{N}^{\mathbb{N}}$ (or equivalently, a continuous image of the irrational numbers in the unit interval).
- the kernel of the $\mathcal{A}$-operation, i.e. A analytic iff

$$A = \bigcup_{\mathbb{N}^{\mathbb{N}}} \bigcap_{k} S_{n_1 \ldots n_k}$$

with $\{S_{n_1 \ldots n_k}\}$ a family of sets indexed with all finite sequences of natural numbers. It is called the "defining system" of A.

Order of growth

Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$

- If f is holomorphic, its order of growth can be defined as,

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Polynomials and rational functions have $\rho = 0$ (and $\sharp \mathrm{As}(f) = 1$). The exponential function has $\rho = 1$ (and $\sharp \mathrm{As}(f) = 2$).

Holomorphic case

The bigger the order of growth, the richer the behavior of f near infinity?

Theorem (Ahlfors)

If f is entire of order ρ then,

$$\sharp \mathrm{As}(f) \leq 2\rho + 1.$$

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If $\mathcal{A}$ is not finite, the order of growth of f is infinite.

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Theorem (Eremenko)

For any $0 \le \rho \le \infty$ there exists a meromorhic function f in $\mathbb{C}$ of order ρ such that

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Moreover, for any increasing function $\psi(r) \nearrow \infty$ *(*$r \to \infty$*) there exists* f *meromorphic in* $\mathbb{C}$ *such that*

$$T(r,f) = O(\psi(r)\log^2 r) \quad \text{and} \quad \mathrm{As}(f) = \mathbb{C} \cup \{\infty\}.$$

Meromorphic case. The remaining case

Theorem (C., Drasin, Granados)

Given an analytic set $\mathcal{A} \in \mathbb{C}$ and given $0 \leq \rho \leq \infty$ there exists a meromorphic function, f, defined in $\mathbb{C}$ of order ρ such that

$$\mathrm{As}(f) = \mathcal{A}.$$

Moreover, for any increasing function $\psi(r) \nearrow \infty$ ($r \to \infty$) there exists f meromorphic in $\mathbb{C}$ such that

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Outline of the proof

WLOG $\mathcal{A}$ contains 0 and ∞.
Assume $A := \mathcal{A} \setminus \{\infty\} \subset D(0,1)$ and ψ are given (case $\rho = 0$).

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2. Approximate the Riesz charge of U by point masses, that will conform the Riesz mass of $\log|g|$, for g meromorphic in $\mathbb{C}$.
3. The approximation is "good" outside a (small) set E, and in E $\log|g|$ is small (large) whenever U is small (large) so $\log|g|$ will "mimic" the behavior of U.

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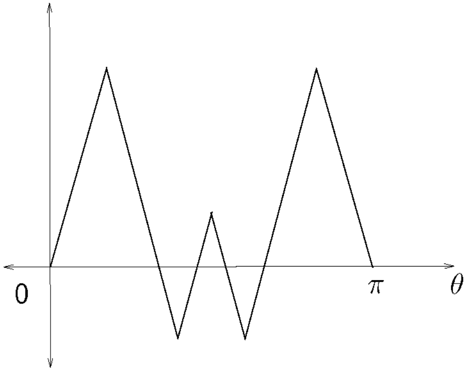
5. Modify g by a quasiconformal map Φ, so asymptotic paths that approach 0 will approach $a \in A$. Now $F = \Phi \circ g$ is quasiregular of dilatation σ_F.
6. Solve the Beltrami equation $\overline{\partial}\phi = \sigma_F \partial\phi$ to find another quasiconformal map so that $f = F \circ \phi$ is meromorphic, $T(r, f) = O(\psi(r) \log^2 r)$ and $\mathrm{As}(f) = \mathcal{A}$.

Sketch of the proof. Construction of U

$U(re^{\theta})$ is a piecewise linear function on θ (for fixed r) so that is symmetric: $U(z) = -U(-z)$ and $U(r) = U(-r) = 0$, therefore we will only show U in the upper-half plane.



Graph of $U(re^{i\theta})$ for fixed r.

Sketch of the proof. Graphs of $U(re^{i\theta})$

- the slope of the linear pieces of $U(re^{i\theta})$ is $L(r)$ for some increasing function $L(r) \nearrow \infty$ as $(r \to \infty)$. The function L depends on ψ,

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In the upper-half plane

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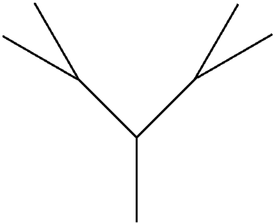

Paths of local minima, Υ.

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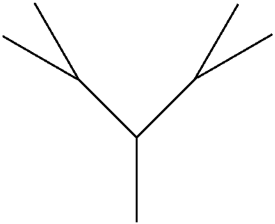
- As the local minima split their paths follow the structure of a dyadic tree.
- The paths of the local maxima separate the branches of the tree and remain undivided.



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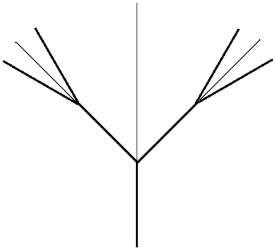

Paths of local minima and maxima.

Sketch of the proof. Riesz mass of U

The Riesz (signed) mass of U is

$$\Delta U = \mu + \mu_e$$

where

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The measure μ_e is "small" and only μ will be approximated by point masses located on the branches of Γ following standard approximation techniques (Yulmukhametov, Liubarskii-Malinnikova...).

Sketch of the proof. The role of the analytic set A

The function U is constructed in such a way that

- $U \to +\infty$ on paths of local maxima,
- $U \to -\infty$ on paths of local minima,

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The rate at which U approaches $\pm\infty$ depends on the analytic set A by associating to each point of A a branch of Υ.

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Example with A a Cantor set

Let A be a Cantor set given by $A = \bigcap_{j=1}^{\infty} E_j$ with $E_j = \bigcup_n Q_n^j$ where,

- Q_n^j are closed cubes in $\mathbb{C}$.
- For every $Q_n^j \subset E_j$ there exists a $Q_m^{j-1} \subset E_{j-1}$ such that $Q_n^j \subset Q_m^{j-1}$.

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Assume further that

- $\mathrm{diam}(Q_n^j) = \delta_j$ for some given sequence $\delta_j \searrow 0$.

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Pick a fixed $a_n^j \in Q_n^j$ for all n and j. Each $a \in A$ is given by $a = \bigcap_j Q_n^j$ so there is a sequence $a_n^j \to a$ $(j \to \infty)$ in such a way that $|a_n^j - a| \leq \delta_j$.

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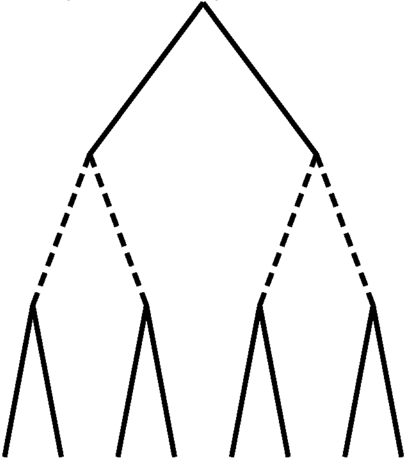
The association of points in A and branches of Υ is done in the obvious way (recall $0 \in A$).



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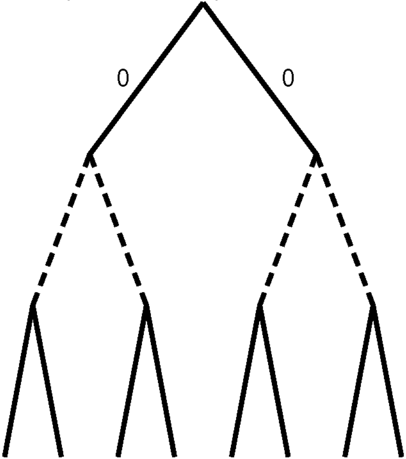

Associate 0 to the first branches, until in generation k, $\sharp\{Q_n^1\} \leq 2^k$ for some $k = k(1)$.

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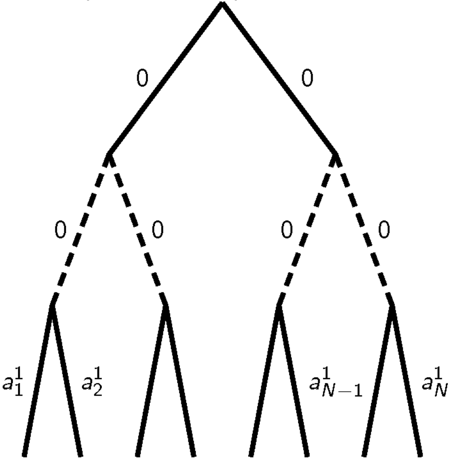

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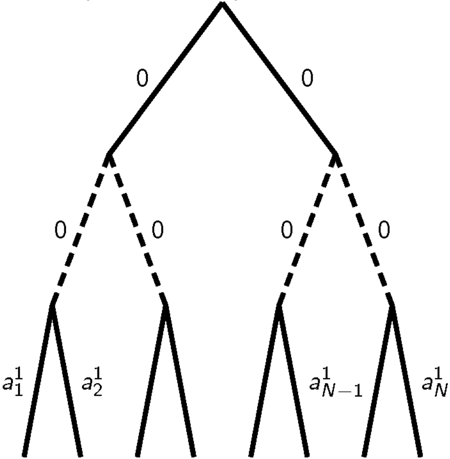

Notice $|a^1_n| \leq \mathrm{diam}(A) \leq 2$

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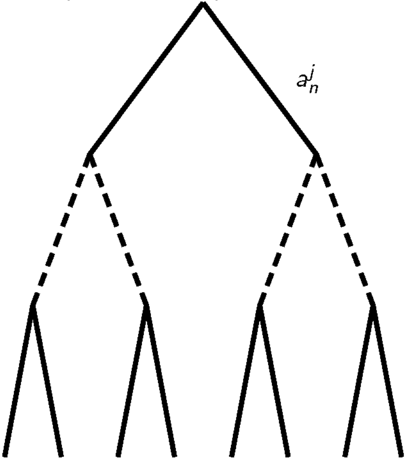

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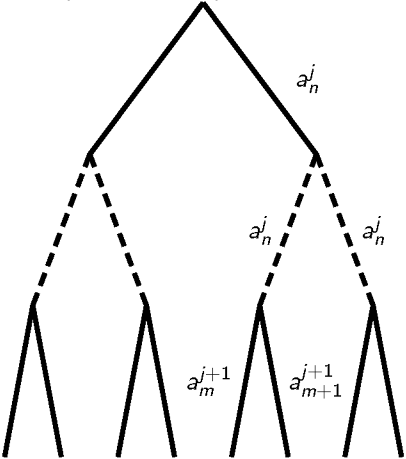

Assume a^j_n is associated to a branch.
Wait until $\sharp\{Q^j_n\} \leq 2^k$ (for all n) for some $k = k(j)$.
Associate to those branches, $a^{j+1}_m \in Q^{j+1}_m \subset Q^j_n$.

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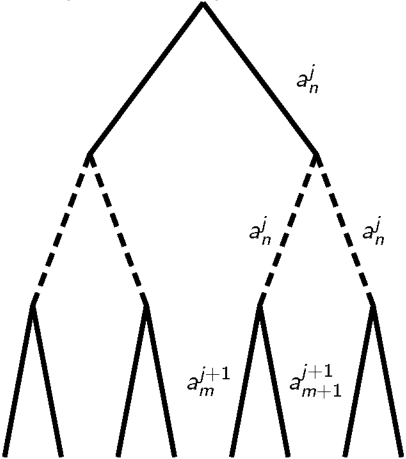

Notice
$|a_n^j - a_m^{j+1}| \leq \mathrm{diam}(Q_n^j) = \delta_j$

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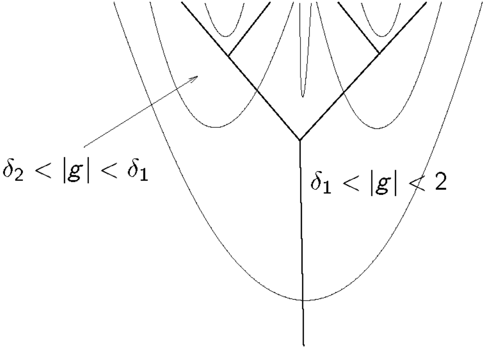

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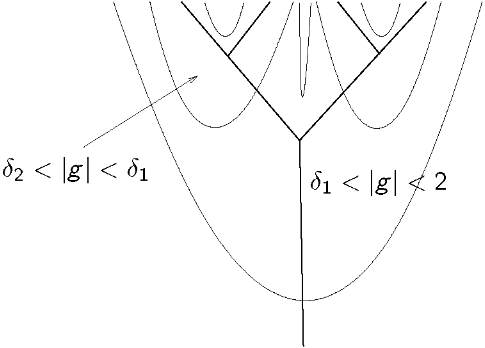

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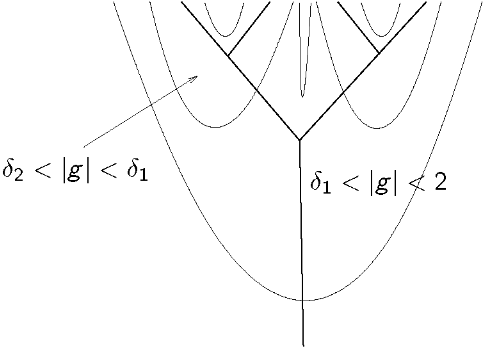

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Lemma

Given $K > 1$ and $R > 0$ there exists $0 < \delta < R$ so that if $|a| < \delta$, then there is a K-quasiconformal map φ so that

$$\varphi(z) = \begin{cases} z, & |z| > R, \\ z + a, & |z| \leq \delta. \end{cases}$$

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Given a sequence $\{K_j\}$ such that $\Pi_j K_j = K < \infty$ use lemma (starting with $R = 2$) to find a sequence $\{\delta_j\}$.

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To "map" the set $\{0, \infty\}$ into $A \cup \{\infty\} = \mathcal{A}$ we compose recursively with quasiconformal translations:

Lemma

Given $K > 1$ and $R > 0$ there exists $0 < \delta < R$ so that if $|a| < \delta$, then there is a K-quasiconformal map φ so that

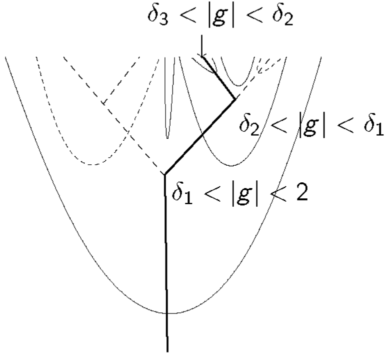
$$\varphi(z) = \begin{cases} z, & |z| > R, \\ z + a, & |z| \leq \delta. \end{cases}$$

Given a sequence $\{K_j\}$ such that $\Pi_j K_j = K < \infty$ use lemma (starting with $R = 2$) to find a sequence $\{\delta_j\}$. This will be our "given" sequence $\{\delta_j\}$.

Sketch of the proof. Composition with qc transformations

Example with A a Cantor set

Given $a \in A$, consider $a^j \to a$ and the associated branch.



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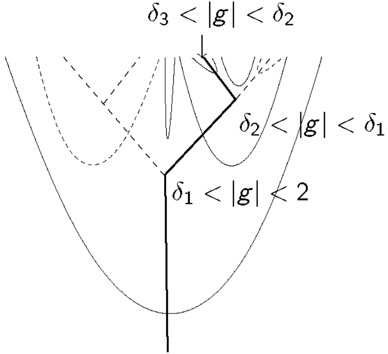

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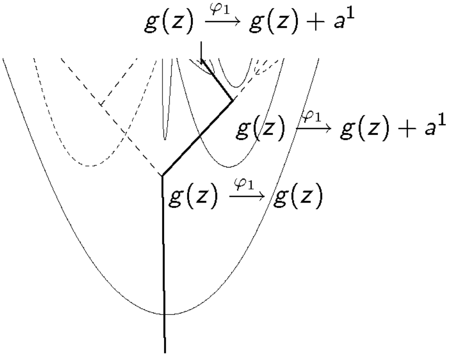

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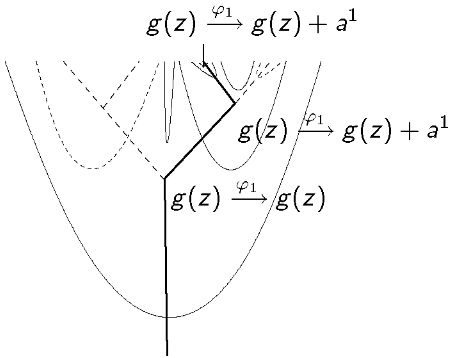

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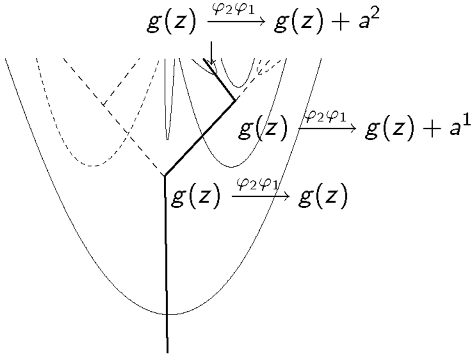

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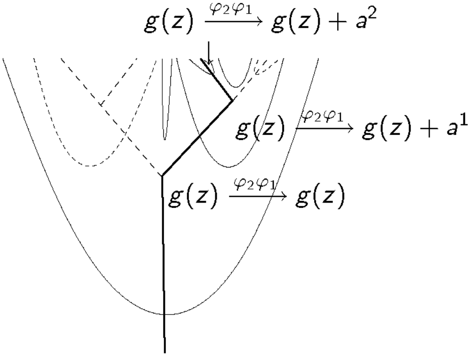

Along this branch, after composing with qc transformations $F = \Phi \circ g$

Sketch of the proof. Final tuning

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- If $A = \mathcal{A} \setminus \{\infty\}$ is unbounded, then decompose A in a countable number of bounded sets of diameter at most 2, and construct countable many dyadic trees in a similar way.

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For each $n_1 \dots n_k$ pick $a_{n_1 \dots n_k} \in S_{n_1 \dots n_k}$.

Each $a \in A$ is given by $a = \bigcap_k S_{n_1 \dots n_k}$ so there is a sequence $\{a_{n_1 \dots n_k}\}$ so that

$$a_{n_1 \dots n_k} \to a \quad (k \to \infty) \text{ and } |a_{n_1 \dots n_k} - a| \leq \delta_k.$$

Sketch of the proof. Analytic sets and binary expansions

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$$\sum_{j=1}^{N} \frac{\xi_j}{2^j} = \sum_{j=1}^{k} \frac{1}{2^{n_1+\cdots+n_j}}$$

Notice that $N = n_1 + \cdots + n_k$.

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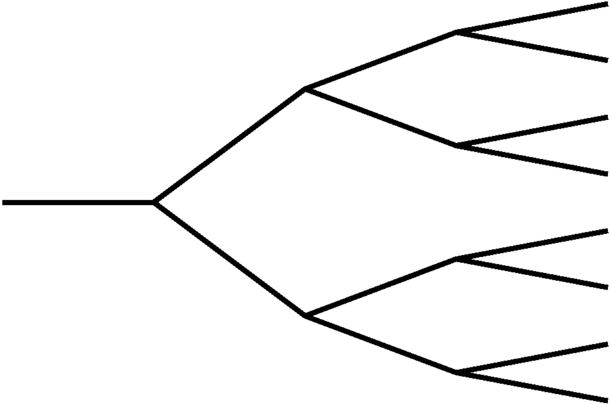
Sketch of the proof. Analytic sets and trees

To $0 \in A$ associate any finite sequence of 0's.



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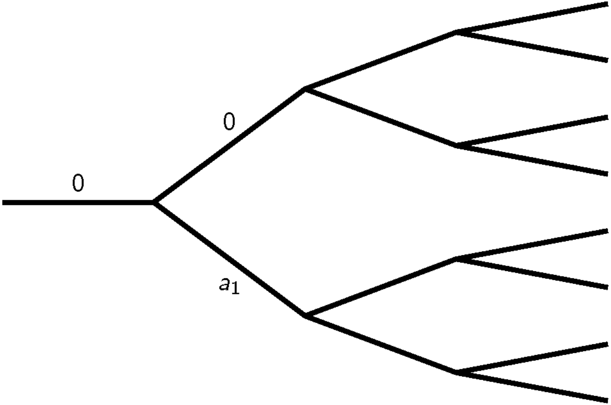

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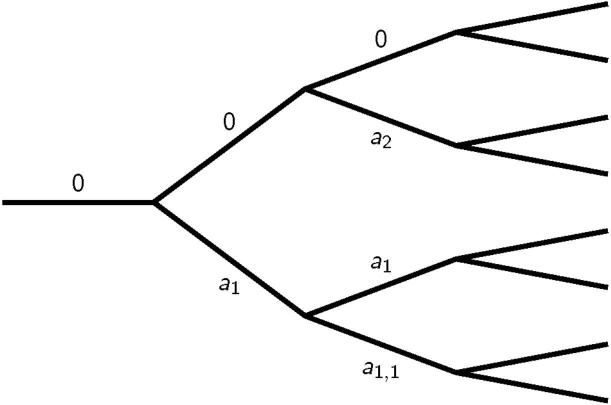

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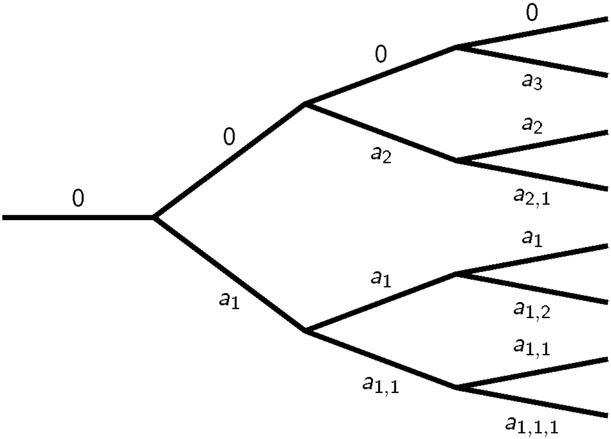

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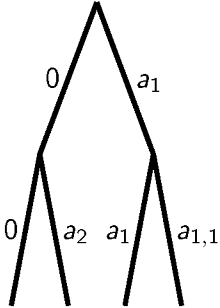
With this association the distance between two adjacent points of consecutive generations does not decrease with the generation!



In the second generation:

- $|a_1 - a_{1,1}| \leq \delta_1$ since $a_1, a_{1,1} \in S_1$
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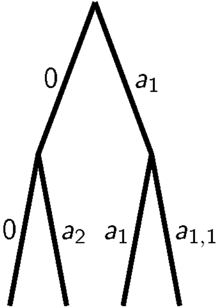

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- $|a_2 - a_{2,1}| \leq \delta_1$ since $a_2, a_{2,1} \in S_2$ and $|a_1 - a_{1,2}| \leq \delta_1$ since $a_1, a_{1,2} \in S_1$
- $|a_3| \leq 2$ since $0, a_3 \in A$

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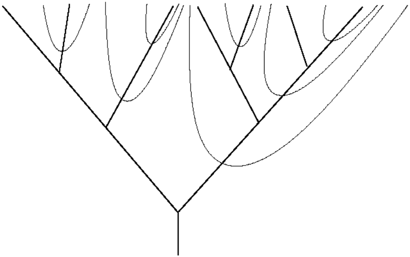

$\square$ $\delta_1 < |g| < 2$

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Now there are branches of the tree where $|g|$ is bounded and some work has to be done to show that g does not take any asymptotic value on those branches to get $\mathrm{As}(g) = \{0, \infty\}$.

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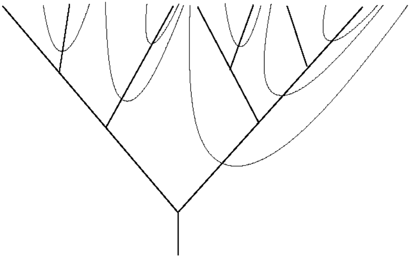

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The end

Thank you!